

# Non-singlet splitting functions in QED

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## Abstract

Iterative solution of QED evolution equations for non-singlet electron structure functions is considered. Analytical expressions in the fourth and fifth orders are presented in terms of splitting functions. Relation to the existing exponentiated solution is discussed. PACS: 12.20.-m Quantum electrodynamics, 12.20.Ds Specific calculations

## 1 Introduction

In this paper we are going to discuss properties of the QED non-singlet splitting functions. Some details of derivation of the fourth and fifth order approximations are given. The related subjects concerning the QED structure functions (SF) themselves are touched very briefly, mainly to show, how do the higher order splitting functions enter into the SF. An extended discussion about the SF can be found in papers [1, 2, 3, 4] and references therein. In Ref. [5] a fifth order perturbative solution for the non-singlet SF was derived within the *ad hoc* exponentiation procedure [6].

The Dokshitzer–Gribov–Lipatov–Altarelli–Parisi evolution equation for the non-singlet electron structure function reads

$$\mathcal{D}^{NS}(z, Q^2) = \delta(1 - z) + \int_{m^2}^{Q^2} \frac{\alpha(q^2)}{2\pi} \frac{dq^2}{q^2} \int_z^1 \frac{dx}{x} P^{(1)}(x) \mathcal{D}^{NS}\left(\frac{z}{x}, q^2\right), \quad (1)$$

where  $m$  is the electron mass;  $P^{(1)}$  is the first order non-singlet splitting function;  $\alpha(q^2)$  is the QED running coupling constant. Here we are going to consider only the electron contribution to vacuum polarization:

$$\alpha(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln \frac{q^2}{m^2}}. \quad (2)$$

It is worth noting that only the one-loop approximation (re-summed) gives the leading log contribution, while higher orders provide only next-to-leading corrections. The account of the

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running coupling constant in Eq. (1) is associated with the leading log radiative corrections due to pair production in the so-called non-singlet channel. Usually pair production is considered separately from the pure photonic correction, because of different conditions of registration. Nevertheless, we are going to evaluate the general equation, and then point out the part, corresponding to pair production. An account of other non-singlet pair contributions due to muons,  $\tau$ -leptons, and hadrons can be done within a certain approximation in the final formulae.

The singlet pair production mechanism in the leading logarithmic approximation is described by the so-called singlet electron structure function  $\mathcal{D}^S$  (see [1, 3, 4]); we are not going to discuss it here.

## 1.1 Splitting function

Function  $P^{(1)}(z)$  in Eq. (1) is the so-called splitting function. It serves as a kernel function in the evolution equation. The function (multiplied by the proper coefficient) gives the first order leading log correction to the probability, which is just  $\mathcal{D}^{NS}(z, Q^2)$ , to find an electron (quark) with energy fraction  $z$  in the initial electron (quark):

$$P^{(1)}(z) \equiv P_e^e(z) = \left( \frac{1+z^2}{1-z} \right)_+ = \frac{1+z^2}{1-z} - \delta(1-z) \int_0^1 dx \frac{1+x^2}{1-x}. \quad (3)$$

A simple derivation of the function can be found in Ref. [7].

The splitting function is a generalised mathematical function. For practical applications we prefer to use the following definition, which allows to avoid explicitly the mutual cancellation of infinite quantities during numerical computations:

$$\begin{aligned} P^{(1)}(z) &= \lim_{\Delta \rightarrow 0} \left\{ 2 \left( \ln \Delta + \frac{3}{4} \right) \delta(1-z) + \frac{1+z^2}{1-z} \Theta(1-z-\Delta) \right\} \\ &\equiv P_{\Delta}^{(1)} \delta(1-z) + P_{\Theta}^{(1)} \Theta(1-z-\Delta). \end{aligned} \quad (4)$$

As could be seen from Eqs. (3,4), the function satisfies the following normalisation condition:

$$\int_0^1 dz P^{(1)}(z) = 0. \quad (5)$$

This condition is a manifestation of the Kinoshita-Lee-Nauenberg theorem [8]: it provides the cancellation of mass singularities.

## 2 Iterative solution

In QED the evolution equation can be solved to any desired order of perturbation theory by means of iteration. The initial approximation for the structure function is just  $\delta(1-z)$ . The first iteration gives

$$\mathcal{D}^{NS}(z, Q^2) = \delta(1-z) + \frac{\beta}{4} P^{(1)}(z) + \mathcal{O}(\alpha^2), \quad \beta = \frac{2\alpha}{\pi}(L-1), \quad L = \ln \frac{Q^2}{m^2}, \quad (6)$$

where  $L$  is the so-called large logarithm.

On the next step of the procedure we need to calculate the integral

$$\int_z^1 \frac{dx}{x} P^{(1)}(x) P^{(1)}\left(\frac{z}{x}\right) \equiv P^{(1)} \otimes P^{(1)}(z). \quad (7)$$

This is a typical Mellin convolution

$$P^{(n+1)}(z) = \int_z^1 \frac{dx}{x} P^{(1)}(x) P^{(n)}\left(\frac{z}{x}\right) = \int_0^1 dx_1 \int_0^1 dx_2 P^{(1)}(x_1) P^{(n)}(x_2) \delta(z - x_1 x_2). \quad (8)$$

In this way step by step we get the solution for the evolution equation to the fifth order:

$$\mathcal{D}^{\text{NS}}(z, Q^2) = \mathcal{D}_\gamma^{\text{NS}}(z, Q^2) + \mathcal{D}_{e^+e^-}^{\text{NS}}(z, Q^2), \quad (9)$$

$$\mathcal{D}_\gamma^{\text{NS}}(z, Q^2) = \delta(1-z) + \sum_{n=1}^5 \frac{1}{n!} \left(\frac{\beta}{4}\right)^n P^{(n)}(z) + \mathcal{O}(\alpha^6), \quad (10)$$

$$\begin{aligned} \mathcal{D}_{e^+e^-}^{\text{NS}}(z, Q^2) &= \frac{1}{3} \left(\frac{\beta}{4}\right)^2 P^{(1)}(z) + \left(\frac{\beta}{4}\right)^3 \left[ \frac{1}{3} P^{(2)} + \frac{4}{27} P^{(1)} \right] + \left(\frac{\beta}{4}\right)^4 \left[ \frac{1}{6} P^{(3)} + \frac{11}{54} P^{(2)} \right. \\ &\quad \left. + \frac{2}{27} P^{(1)} \right] + \left(\frac{\beta}{4}\right)^5 \left[ \frac{1}{18} P^{(4)} + \frac{7}{54} P^{(3)} + \frac{10}{81} P^{(2)} + \frac{16}{405} P^{(1)} \right] + \mathcal{O}(\alpha^6). \end{aligned} \quad (11)$$

We denoted by index  $\gamma$  the pure photonic part of the SF. The other part describes pair corrections, and, starting from the third order, with possible simultaneous photon radiation. Recently we considered the numerical impact of the higher order pair corrections to electron-positron annihilation in paper [9].

At the level of the non-singlet structure function the condition (5) reads

$$\int_0^1 dz \mathcal{D}^{\text{NS}}(z, Q^2) = 1, \quad \int_0^1 dz P^{(n)}(z) = 0, \quad n = 1, 2, \dots \quad (12)$$

This condition has also a trivial probabilistic meaning: the sum of probabilities of all allowed emission processes is unit. In other words, one is always able to find an electron in the initial electron.

Now we return to the splitting function properties. Using prescription (4) we can represent the integral of the product of our two generalised functions in the form

$$\begin{aligned} P^{(n+1)}(z) &= P_\Delta^{(n+1)} \delta(1-z) + P_\Theta^{(n+1)} \Theta(1-z-\Delta), \quad \Delta \rightarrow 0, \\ P_\Theta^{(n+1)}(z) &= P_\Theta^{(1)}(z) P_\Delta^{(n)} + P_\Delta^{(1)} P_\Theta^{(n)}(z) + \int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x} P_\Theta^{(1)}(x) P_\Theta^{(n)}\left(\frac{z}{x}\right). \end{aligned} \quad (13)$$

The  $\Delta$ -part of the splitting function can be obtained from the condition (5,12):

$$P_\Delta^{(n+1)} = - \int_1^{1-\Delta} dz P_\Theta^{(n+1)}(z). \quad (14)$$

Instead of the above trick we can use, as was discussed in [1], the known solution of the evolution equation in the soft limit [10]:

$$\mathcal{D}_\gamma^{\text{NS}}(z, Q^2) \Big|_{z \rightarrow 1} = \frac{\beta}{2} \frac{(1-z)^{\beta/2-1}}{\Gamma(1+\beta/2)} \exp \left\{ \frac{\beta}{2} \left( \frac{3}{4} - C \right) \right\}, \quad (15)$$

where  $C$  is the Euler constant ( $C \approx 0.57721566$ ). In order to obtain the  $\Delta$ -part of a splitting function of the desired order we have to integrate Eq. (15) over the interval  $1 - \Delta < z < 1$ , and then expand into a series in  $\alpha$ :

$$\int_{1-\Delta}^1 dz \mathcal{D}_\gamma^{\text{NS}}(z, Q^2) = \exp \left\{ \frac{\beta}{2} \ln \Delta + \frac{3\beta}{8} \right\} \frac{\exp(-C\beta/2)}{\Gamma(1+\beta/2)}. \quad (16)$$

Now we have to expand the exponent and use also the following formula (see Appendix):

$$\begin{aligned} \frac{\exp(-C\beta/2)}{\Gamma(1+\beta/2)} &= 1 - \frac{1}{2} \left( \frac{\beta}{2} \right)^2 \zeta(2) + \frac{1}{3} \left( \frac{\beta}{2} \right)^3 \zeta(3) + \frac{1}{16} \left( \frac{\beta}{2} \right)^4 \zeta(4) \\ &+ \frac{1}{5} \left( \frac{\beta}{2} \right)^5 \zeta(5) - \frac{1}{6} \left( \frac{\beta}{2} \right)^5 \zeta(2)\zeta(3) + \mathcal{O}(\beta^6). \end{aligned} \quad (17)$$

The second and third order splitting functions are well known (see paper [1] and references therein) and used in many various applications. For the sake of completeness we put here the expressions:

$$\begin{aligned} P_\Theta^{(2)}(z) &= 2 \left[ \frac{1+z^2}{1-z} \left( 2 \ln(1-z) - \ln z + \frac{3}{2} \right) + \frac{1+z}{2} \ln z - 1 + z \right], \\ P_\Delta^{(2)} &= 4 \left( \ln \Delta + \frac{3}{4} \right)^2 - 4\zeta(2), \\ P_\Theta^{(3)}(z) &= 24 \frac{1+z^2}{1-z} \left( \frac{1}{2} \ln^2(1-z) + \frac{3}{4} \ln(1-z) - \frac{1}{2} \ln z \ln(1-z) + \frac{1}{12} \ln^2 z \right. \\ &\quad \left. - \frac{3}{8} \ln z + \frac{9}{32} - \frac{1}{2} \zeta(2) \right) + 6(1+z) \ln z \ln(1-z) - 12(1-z) \ln(1-z) \\ &\quad + \frac{3}{2} (5-3z) \ln z - 3(1-z) - \frac{3}{2} (1+z) \ln^2 z + 6(1+z) \text{Li}_2(1-z), \\ P_\Delta^{(3)} &= 8 \left( \ln \Delta + \frac{3}{4} \right)^3 - 24\zeta(2) \left( \ln \Delta + \frac{3}{4} \right) + 16\zeta(3). \end{aligned} \quad (18)$$

The definition of the Riemann  $\zeta$ -functions, dilogarithm, and other special functions are given in Appendix.

By means of the convolution procedure (8,13) we found

$$\begin{aligned} P_\Theta^{(4)}(z) &= 144 \left\{ \frac{1+z^2}{1-z} \left[ \frac{2}{9} \ln^3(1-z) + \frac{1}{2} \ln^2(1-z) + \left( \frac{3}{8} - \frac{2}{3} \zeta(2) \right) \ln(1-z) \right. \right. \\ &\quad \left. \left. - \frac{1}{3} \ln^2(1-z) \ln z + \frac{1}{9} \ln(1-z) \ln^2 z - \frac{1}{2} \ln(1-z) \ln z - \frac{\ln^3 z}{108} + \frac{\ln^2 z}{12} \right. \right. \\ &\quad \left. \left. + \left( \frac{\zeta(2)}{3} - \frac{3}{16} \right) \ln z - \frac{1}{9} \ln z \text{Li}_2(1-z) - \frac{2}{9} \text{S}_{1,2}(1-z) + \frac{3}{32} - \frac{\zeta(2)}{2} + \frac{4}{9} \zeta(3) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& - \frac{1-z}{3} \ln^2(1-z) - \frac{1-z}{6} \ln(1-z) + \frac{1+z}{6} \ln^2(1-z) \ln z \\
& - \frac{1+z}{12} \ln(1-z) \ln^2 z + \frac{5-3z}{12} \ln(1-z) \ln z + \frac{7(1+z)}{864} \ln^3 z + \frac{5z-11}{144} \ln^2 z \\
& + \left( \frac{43-5z}{288} - \frac{1+z}{6} \zeta(2) \right) \ln z + \frac{1+z}{3} \ln(1-z) \text{Li}_2(1-z) - \frac{1+z}{3} \text{Li}_3(1-z) \\
& + \frac{1+z}{6} \text{S}_{1,2}(1-z) + \frac{1+z}{12} \text{Li}_2(1-z) - \frac{11(1-z)}{144} + \frac{1-z}{3} \zeta(2) \Big\}, \tag{20}
\end{aligned}$$

$$P_{\Delta}^{(4)} = 16 \left( \ln \Delta + \frac{3}{4} \right)^4 - 96 \zeta(2) \left( \ln \Delta + \frac{3}{4} \right)^2 + 128 \zeta(3) \left( \ln \Delta + \frac{3}{4} \right) + 24 \zeta(4). \tag{21}$$

At the next step we got

$$\begin{aligned}
P_{\Theta}^{(5)}(z) = & 720 \left\{ \frac{1+z^2}{1-z} \left[ \frac{1}{9} \ln^4(1-z) + \frac{1}{3} \ln^3(1-z) + \left( \frac{3}{8} - \frac{2}{3} \zeta(2) \right) \ln^2(1-z) + \left( \frac{3}{16} \right. \right. \right. \\
& - \left. \zeta(2) + \frac{8}{9} \zeta(3) \right) \ln(1-z) - \frac{2}{9} \ln z \ln^3(1-z) + \left( \frac{1}{9} \ln^2 z - \frac{1}{2} \ln z \right) \ln^2(1-z) \\
& + \left( -\frac{1}{54} \ln^3 z + \frac{1}{6} \ln^2 z - \left( \frac{2}{9} \text{Li}_2(1-z) + \frac{3}{8} - \frac{2}{3} \zeta(2) \right) \ln z - \frac{4}{9} \text{S}_{1,2}(1-z) \right) \ln(1-z) \\
& + \frac{1}{1080} \ln^4 z - \frac{1}{72} \ln^3 z + \left( \frac{1}{16} - \frac{1}{9} \zeta(2) + \frac{1}{18} \text{Li}_2(1-z) \right) \ln^2 z + \left( -\frac{3}{32} + \frac{1}{2} \zeta(2) \right. \\
& - \left. \frac{4}{9} \zeta(3) + \frac{1}{9} \text{S}_{1,2}(1-z) - \frac{1}{6} \text{Li}_2(1-z) + \frac{2}{9} \text{Li}_3(1-z) \right) \ln z + \frac{1}{9} \text{Li}_2^2(1-z) \\
& - \left. \frac{1}{3} \text{S}_{1,2}(1-z) + \frac{9}{256} - \frac{3}{8} \zeta(2) + \frac{2}{3} \zeta(3) + \frac{1}{6} \zeta(4) \right] + \left( \frac{(1+z)}{9} \ln z \right. \\
& - \left. \frac{2(1-z)}{9} \right) \ln^3(1-z) + \left( -\frac{(1+z)}{12} \ln^2 z + \frac{5-3z}{12} \ln z + \frac{(1+z)}{3} \text{Li}_2(1-z) \right. \\
& - \left. \frac{1-z}{6} \right) \ln^2(1-z) + \left( \frac{7(1+z)}{432} \ln^3 z + \frac{5z-11}{72} \ln^2 z + \frac{43-5z}{144} \ln z \right. \\
& - \left. \frac{(1+z)}{3} \zeta(2) \ln z + \frac{(1+z)}{6} \text{Li}_2(1-z) + \frac{(1+z)}{3} \text{S}_{1,2}(1-z) - \frac{2(1+z)}{3} \text{Li}_3(1-z) \right. \\
& + \left. \frac{2(1-z)}{3} \zeta(2) - \frac{11(1-z)}{72} \right) \ln(1-z) - \frac{(1+z)}{1152} \ln^4 z + \frac{23-9z}{1728} \ln^3 z \\
& + \left( \frac{(1+z)}{12} \zeta(2) - \frac{33+5z}{576} - \frac{5(1+z)}{144} \text{Li}_2(1-z) \right) \ln^2 z + \left( \frac{(1-z)}{9} \text{Li}_2(1-z) \right. \\
& - \left. \frac{(1+z)}{8} \text{S}_{1,2}(1-z) + \frac{49-39z}{576} + \frac{2(1+z)}{9} \zeta(3) - \frac{5-3z}{12} \right) \ln z + \left( \frac{19(1+z)}{144} \right. \\
& - \left. \frac{(1+z)}{3} \zeta(2) \right) \text{Li}_2(1-z) + \frac{11-5z}{36} \text{S}_{1,2}(1-z) - \frac{(1+z)}{6} \text{Li}_3(1-z) \\
& + \frac{2(1+z)}{3} \text{Li}_4(1-z) - \frac{(1+z)}{3} \text{S}_{2,2}(1-z) - \frac{5(1+z)}{72} \text{S}_{1,3}(1-z) \\
& - \left. \frac{5(1-z)}{288} + \frac{(1-z)}{6} \zeta(2) - \frac{4(1-z)}{9} \zeta(3) \right\}, \tag{22}
\end{aligned}$$

$$\begin{aligned}
P_{\Delta}^{(5)} &= 32\left(\ln \Delta + \frac{3}{4}\right)^5 - 320\zeta(2)\left(\ln \Delta + \frac{3}{4}\right)^3 + 640\zeta(3)\left(\ln \Delta + \frac{3}{4}\right)^2 \\
&+ 240\zeta(4)\left(\ln \Delta + \frac{3}{4}\right) + 768\zeta(5) - 640\zeta(2)\zeta(3).
\end{aligned} \tag{23}$$

### 3 Conclusions

Table 1: Integral  $I(x)$  in different approximations.

$x$	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha^2)$	$\mathcal{O}(\alpha^3)$	$\mathcal{O}(\alpha^4)$	$\mathcal{O}(\alpha^5)$	exponent.
0.01	0.99972722	0.99970278	0.99970216	0.99970216	0.99970216	0.99970216
0.1	0.99713067	0.99696386	0.99696237	0.99696241	0.99696241	0.99696241
0.5	0.97933805	0.97871205	0.97873132	0.97873149	0.97873148	0.97873148
0.9	0.91043156	0.91243286	0.91253629	0.91252777	0.91252803	0.91252802
0.99	0.79019566	0.80982514	0.80884689	0.80886291	0.80886436	0.80886422
0.999	0.66569598	0.71915664	0.71380222	0.71416672	0.71415030	0.71415065

The expressions obtained for the fourth and fifth order splitting functions were checked to satisfy the condition (12). In this way we see the agreement between the iteration procedure for  $P_{\Theta}^{(n)}(z)$  and the expansion of the known solution for  $P_{\Delta}^{(n)}$ . Our result does also coincide with the corresponding expansion of the exponentiated solution from Ref. [5]. So, we reproduced the known result, but in a different approach; and the higher order splitting functions are given explicitly. The exponentiation and order-by-order calculations are complementary to each other. As could be seen from Table 1, the numerical difference between the exponentiated and non-exponentiated results is negligible, and one may choose safely the approach, which he likes. The results of our paper can be used to estimate higher order radiative correction and to analyse the numerical difference between exponentiated and order-by-order calculations. Here we should note, that in a realistic situation in order to provide a high theoretical precision one should take into account also sub-leading radiative corrections, which can be obtained only by direct perturbative calculations.

In Table 1 the values of integral

$$I(x) = \int_x^1 dz \mathcal{D}_{\gamma}^{\text{NS}}(z, Q^2) \tag{24}$$

of the pure photonic part of the non-singlet SF is given for different order approximations for  $Q^2 = 10^4 \text{ GeV}^2$ ,  $L \approx 24.37$ . For the last column the exponentiated result is obtained by using formula (11) from Ref. [5]. In Table 2 we present the corresponding values of integrals of the splitting functions themselves:

$$J^{(n)}(x) = \int_x^1 dz P^{(n)}(z). \tag{25}$$

In the last line of Table 2 we put also the values of the corresponding  $\Delta$ -parts. Note, that in reality, to simulate the limit  $\Delta \rightarrow 0$ , and so to eliminate the dependence on  $\Delta$  of numbers for integrals  $I(x)$  and  $J^{(n)}(x)$ , we used  $\Delta = 10^{-10}$ .

Table 2: Integrals of splitting functions  $J^{(n)}(x)$  and  $P_{\Delta}^{(n)}$ .

	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha^2)$	$\mathcal{O}(\alpha^3)$	$\mathcal{O}(\alpha^4)$	$\mathcal{O}(\alpha^5)$
$x$	$J^{(n)}(x)$				
0.01	-0.0101	-0.0664	-0.1854	-0.1409	0.5232
0.1	-0.1057	-0.4529	-0.4487	1.6579	2.7010
0.5	-0.7613	-1.6997	5.7829	7.5130	-99.1465
0.9	-3.3002	5.4338	31.0395	-376.7425	2102.5492
0.99	-7.7303	53.2969	-293.5939	708.7696	11810.0253
0.999	-12.3175	145.1533	-1606.9698	16122.5016	-133803.0981
$\Delta$	$P_{\Delta}^{(n)}$				
0.001	-12.3155	145.0921	-1605.5843	16095.0956	-133303.9430

## Appendix A

The Riemann  $\zeta$ -functions are defined as usual:

$$\begin{aligned}\zeta(n) &= \sum_{k=1}^{\infty} \frac{1}{k^n}, & \zeta(2) &= \frac{\pi^2}{6}, & \zeta(3) &\approx 1.20205690315959, \\ \zeta(4) &= \frac{\pi^4}{90}, & \zeta(5) &\approx 1.03692775514337.\end{aligned}\tag{A.1}$$

Here we define the polilogarithms, which enter into our formulae. We follow the notations of Ref. [11, 12]. The general Nielsen's polilogarithm is

$$S_{n,m}(z) = \frac{(-1)^{n+m-1}}{(n-1)!m!} \int_0^1 \frac{dx}{x} \ln^{n-1}(x) \ln^m(1-xz),\tag{A.2}$$

in particular

$$\begin{aligned}\text{Li}_2(z) &= S_{1,1}(z) = -\int_0^1 \frac{dx}{x} \ln(1-xz), & S_{1,2}(z) &= \frac{1}{2} \int_0^1 \frac{dx}{x} \ln^2(1-xz), \\ \text{Li}_3(z) &= S_{2,1}(z) = \int_0^1 \frac{dx}{x} \ln(x) \ln(1-xz) = \int_0^1 \frac{dx}{x} \text{Li}_2(x), \\ \text{Li}_4(z) &= S_{3,1}(z) = -\frac{1}{2} \int_0^1 \frac{dx}{x} \ln^2(x) \ln(1-xz), & S_{1,3}(z) &= -\frac{1}{6} \int_0^1 \frac{dx}{x} \ln^3(1-xz), \\ S_{2,2}(z) &= -\frac{1}{2} \int_0^1 \frac{dx}{x} \ln(x) \ln^2(1-xz).\end{aligned}\tag{A.3}$$

In order to get expansion (17) it is convenient to use the following representation for  $\Gamma$ -function:

$$\frac{1}{\Gamma(z)} = ze^{Cz} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.\tag{A.4}$$

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